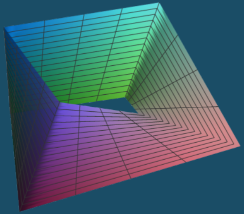

Signature matrices of membranes

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Membranes

A *membrane* is a continuous map $X : [0,1]^2 \to \mathbb{R}^d$ such that all coordinate functions X_i are piecewise[1] continuously differentiable.

Example

[1] on a rectangular partition of $[0,1]^2$

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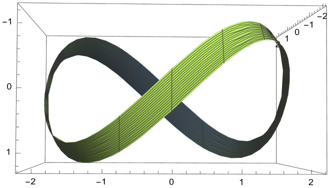

$$(s,t) \mapsto (s\cos(2\pi t)^2 + \sin(4\pi t), 2\cos(2\pi t) - s\sin(2\pi t), 2\sin(2\pi t))$$

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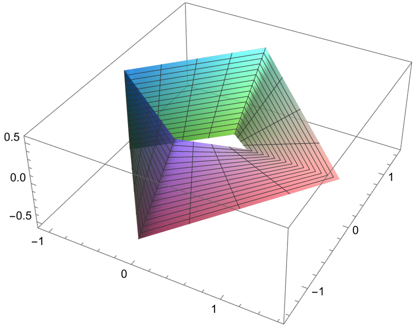

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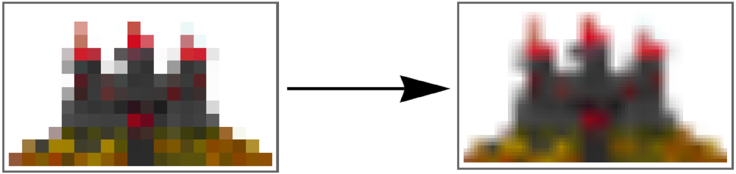


[1] on a rectangular partition of $[0,1]^2$

The signature of a membrane

Definition (The (id-)signature[2])

Let X be a membrane in $\mathbb{R}^d$. Let $A := \mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d} \rangle$ denote the free associative algebra over the letters $\mathtt{1}, \ldots, \mathtt{d}$.

Then the **(id-)signature** of X is the linear form

$$\begin{aligned} \sigma(X) &: A \to \mathbb{R}, \\ \mathtt{i}_1 \ldots \mathtt{i}_k &\mapsto \int_{(\mathbf{s},\mathbf{t}) \in \Delta_k^2} \partial_{12} X_{i_1}(s_1, t_1) \ldots \partial_{12} X_{i_k}(s_k, t_k) \, \mathrm{d}\mathbf{s} \mathrm{d}\mathbf{t} \end{aligned}$$

where $\partial_{12} := \frac{\partial^2}{\partial s \partial t}$, $\mathrm{d}\mathbf{t} = \mathrm{d}t_1 \ldots \mathrm{d}t_k$, $\mathrm{d}\mathbf{s} = \mathrm{d}s_1 \ldots \mathrm{d}s_k$.

[2] J. Diehl, K. Ebrahimi-Fard, F. N. Harang, and S. Tindel. "On the signature of an image". In: *Stochastic Processes and their Applications* (2025), p. 104661.

The signature matrix

Restricting to words of length 2, $\sigma(X)$ defines a $d \times d$ matrix $S(X)$, which we call the *signature matrix* of X:

$$S(X)_{i,j} := \sigma(X)(\texttt{ij})$$

Explicitly, $S(X)_{i,j}$ is given by

$$\int_0^1 \int_0^1 \int_0^{s_2} \int_0^{t_2} \partial_{12} X_i(s_1, t_1) \cdot \partial_{12} X_j(s_2, t_2) \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}t_1 \mathrm{d}t_2$$

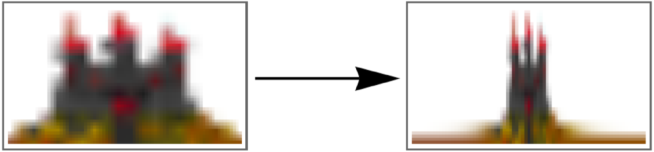
Example

Let $X : [0,1]^2 \to \mathbb{R}^2, (s,t) \mapsto (2st, 3st)$. The signature matrix is

$$\begin{pmatrix} \frac{1}{2} \end{pmatrix}^2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}.$$

Observations

- The signature is invariant under addition of functions that are linear in (s, t) and under reparametrisation.



Observations

- The signature is invariant under addition of functions that are linear in (s,t) and under reparametrisation.
- The signature is equivariant under linear transformations: if X is a membrane in $\mathbb{R}^d$ and A is a $e \times d$ matrix then

$$\sigma(A \circ X)(\mathtt{w}) = \sigma(X)(A^\top.\mathtt{w}).$$

- The signature commutes with tensor products of paths: if X and Y are paths and Z denotes the membrane $(s,t) \mapsto X(s) \otimes Y(t)$, then

$$\sigma(Z) = \sigma(X) \otimes \sigma(Y).$$

  In particular, if $Z(s,t) = X(s) \cdot t$ then $\sigma^{(k)}(Z) = \frac{1}{k!}\sigma^{(k)}(X)$.
- The signature can vanish even for functions that are not linear in (s,t).
- The signature is *not* a homomorphism for the shuffle product on $\mathbb{R}\langle \mathtt{1}, \dots, \mathtt{d} \rangle$.

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Signature matrices of membranes

Theorem (L., Schmitz '24)

Let $d \in \mathbb{N}$. There are no algebraic relations on the set of signature matrices $S(X)$ where X runs over membranes in $\mathbb{R}^d$.

In fact, this holds true already for the signature matrices of *polynomial* or *piecewise bilinear* membranes.

Remark

In contrast, signature matrices of paths always satisfy algebraic relations: they are the zero set of the 2-minors of the symmetric part of a $d \times d$ matrix.

A tale of two membranes

- Studying the algebraic relations on a set leads us to the realm of Algebraic Geometry.
- Inspired by the approach of Améndola, Friz and Sturmfels for path signatures[3], we consider two parametrizable families of membranes for which the signature is a polynomial in the parameters:
  - A *polynomial membrane* of order m, n is a membrane X that is polynomial in both s and t with bidegree $\leq (m, n)$.
  - A *piecewise bilinear membrane* of order m, n is a membrane X such that there are $s_0 = 0 \leq s_1 \leq \cdots \leq s_m = 1$ and $t_0 = 0 \leq t_1 \leq \cdots \leq t_n = 1$ with X biaffine on all squares $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$.
- These types of membranes are also important in application, for example to interpolate discrete data.

[3] C. Améndola, P. Friz, and B. Sturmfels. "Varieties of signature tensors". In: *Forum of Mathematics, Sigma* 7 (2019).

A polynomial membrane

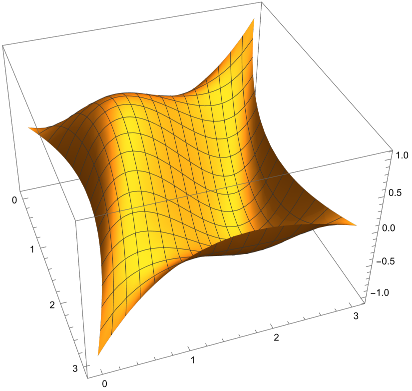

Figure: A Bezier surface with 4×4 control points is a polynomial membrane of order $(3, 3)$.

A piecewise bilinear membrane

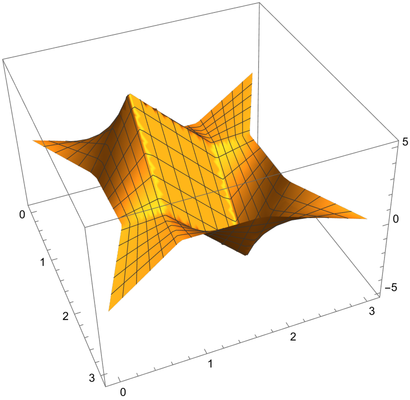

Figure: A piecewise bilinear membrane with 4×4 control points, that is, of order $(3, 3)$.

A tale of two membranes

- Both polynomial membranes of order (m, n) and piecewise bilinear membranes of order (m, n) in $\mathbb{R}^d$ can be uniquely described by $m \times n$-matrices $A_1, \ldots, A_d$.

Example

The matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ corresponds to the polynomial

$$a_{11}st + a_{12}st^2 + a_{21}s^2t + a_{22}s^2t^2$$

and the piecewise bilinear membrane with 9 squares satisfying $X(s, 0) = X(0, t) = 0$ and

$$\begin{pmatrix} X(\frac{1}{2}, \frac{1}{2}) & X(\frac{1}{2}, 1) \\ X(1, \frac{1}{2}) & X(1, 1) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} + a_{11} \\ a_{21} + a_{11} & a_{11} + a_{12} + a_{21} + a_{22} \end{pmatrix}$$

Aside: computing signatures of bilinear membranes

Proposition (L., Schmitz '24)

Given $m \times n$ matrices $A_1, \ldots, A_d$, the k-th level signature of the associated piecewise bilinear membrane in $\mathbb{R}^d$ can be computed in $O(d^k k^3 mn)$.

```
... word = list(np.random.randint(1, d+1, k)) #creates a random word of length k, with letters in\
 {1,...,d}
... wordstr = ''.join(str(x) for x in word)
...
... start_time = timeit.default_timer()
... out = sig(membrane, word)
... elapsed_time = timeit.default_timer() - start_time
...
... # Print the output and elapsed time
... print(f"Evaluated signature of bilinear membrane of order {(m,n)} at word {wordstr} in {elaps\
ed_time:.6f} seconds. Result: {out}")
...
Evaluated signature of bilinear membrane of order (100, 100) at word 21122 in 0.464673 seconds. Result: -115
6190.415377495
>>>
```

Dictionaries for membranes

Let $m, n \in \mathbb{N}$. We can view a collection of $m \times n$ matrices $A_1, \ldots, A_d$ as a map $A : \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^d$.

Lemma

There are membranes $\mathsf{Mon}^{m,n}$ and $\mathsf{Axis}^{m,n}$ in $\mathbb{R}^m \otimes \mathbb{R}^n$ such that for any $A : \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^d$, $A \circ \mathsf{Mon}^{m,n}$ is the polynomial resp. piecewise bilinear membrane represented by A.

We call these membranes *dictionaries*[4] for polynomial resp. piecewise bilinear membranes of order (m, n), and their signature matrices the *core matrices*.

[4] M. Pfeffer, A. Seigal, and B. Sturmfels. "Learning Paths from Signature Tensors". In: *SIAM Journal on Matrix Analysis and Applications* (2019).

A tale of one membrane

- recall *equivariance*: $S(A \circ X) = AS(X)A^\top$
- $\Rightarrow$ the sets of signature matrices we are interested in are given by congruence orbits of the core matrices:

$$AS(\mathsf{Mon}^{m,n})A^\top \quad \text{and} \quad AS(\mathsf{Axis}^{m,n})A^\top.$$

Proposition

The sets of signature matrices for polynomial membranes of order (m, n) and piecewise bilinear membranes of order (m, n) in $\mathbb{R}^d$ agree.

A tale of one membrane

Proof.

- It suffices to find an invertible matrix that takes one core matrix to the other under congruence
- The analogous problem for Mom^m and Axis^m, the dictionaries for polynomial paths of degree $\leq m$ resp. piecewise linear paths with $\leq m$ segments, was solved by Améndola, Friz and Sturmfels.[5]
- Recall: if $Z(s,t) = X(s) \otimes Y(t)$ for paths X, Y then $S(X \otimes Y) = S(X) \otimes S(Y)$.
- Observation: $\mathsf{Mon}^{m,n}(s,t) = \mathsf{Mon}^m(s) \otimes \mathsf{Mon}^n(t)$ and $\mathsf{Axis}^{m,n} = \mathsf{Axis}^m(s) \otimes \mathsf{Axis}^n(t)$.

$\square$

[5] C. Améndola, P. Friz, and B. Sturmfels. "Varieties of signature tensors". In: *Forum of Mathematics, Sigma* 7 (2019).

Matrix varieties from membrane signatures

- Let $\mathcal{M}^{\mathrm{im}}_{d,m,n}$ denote the set of signature matrices of polynomial membranes of order (m, n).
- We are interested in the algebraic relations on $\mathcal{M}^{\mathrm{im}}_{d,m,n}$: so we might as well consider the *Zariski closure* $\mathcal{M}_{d,m,n}$ of $\mathcal{M}^{\mathrm{im}}_{d,m,n}$. This is the set of all points satisfying the relations that hold on $\mathcal{M}^{\mathrm{im}}_{d,m,n}$. We call $\mathcal{M}_{d,m,n}$ the *signature matrix variety*.
- Question: what is the dimension and degree of $\mathcal{M}_{d,m,n}$? Can we understand its ideal?
- Idea: use *normal form under congruence* of the core matrix.[6][7]

[6] R. A. Horn and V. V. Sergeichuk. "Canonical forms for complex matrix congruence and *congruence". In: *Linear Algebra and its Applications* 416.2 (2006).

[7] F. De Terán and F. M. Dopico. "The solution of the equation $XA + AX^\top = 0$ and its application to the theory of orbits". In: *Linear Algebra and its Applications* 434.1 (2011).

Main results

Theorem

Let $\mathcal{S}_{a,b}$ be the variety of $d \times d$ matrices whose symmetric part has rank $\leq a$ and whose skew-symmetric part has rank $\leq b$. Then

1. $\mathcal{S}_{(m-2)(n-2),m+n-2} \subseteq \mathcal{M}_{d,m,n} \subseteq \mathcal{S}_{mn,m+n-2}$ *if m, n are even,*
2. $\mathcal{S}_{2,n-1} \subseteq \mathcal{M}_{d,2,n} \subseteq \mathcal{S}_{2(n-1)+1,n+1}$ *if n is odd,*
3. $\mathcal{S}_{(m-2)(n-1)-1,m+n-1} \subseteq \mathcal{M}_{d,m,n} \subseteq \mathcal{S}_{m(n-1)+1,m+n-1}$ *if $m \geq 4$ is even, n is odd,*
4. $\mathcal{S}_{(m-1)(n-1)+1,m+n-2} = \mathcal{M}_{d,m,n}$ *if m, n are odd.*

Theorem (L., Schmitz '24)

We have $\mathcal{M}_{d,m,n} = \mathbb{R}^{d \times d}$ for $m + n \geq d + 1$.

Dimension and degree?

Theorem (L., Schmitz '24)

For $mn \leq d$ the dimension of $\mathcal{M}_{d,m,n}$ is

- $dmn - \frac{1}{2}m^2n^2 + m^2(n-1) + (m-1)n^2 - \frac{7}{2}mn + 4(m+n) - 4$ *for even m, n.*
- $dmn - \frac{1}{2}m^2n^2 + m^2(n-1) + (m-1)n^2 - \frac{3}{2}mn + m + n$ *for even m, odd n.*
- $dmn - \frac{1}{2}m^2n^2 + m^2(n-1) + (m-1)n^2 - \frac{7}{2}mn + 3(m+n) - 2$ *for odd m, n.*

Theorem (L., Schmitz '24)

For m, n odd, $m + n \leq d$, $\mathcal{M}_{d,m,n}$ is the complete intersection of the vanishing locus of the $((m-1)(n-1)+2))$-minors of the symmetric part with the vanishing locus of the $(m+n)$-Pfaffians of the skew-symmetric part of a $d \times d$ matrix. The degree of $\mathcal{M}_{d,m,n}$ is

$$\frac{1}{2^{d-(m+n)+2}} \prod_{\alpha=0}^{d-(m-1)(n-1)} \frac{\binom{d+\alpha}{d-(m-1)(n-1)+1-\alpha}}{\binom{2\alpha+1}{\alpha}} \prod_{\alpha=0}^{d-(m+n)} \frac{\binom{d+\alpha}{d-(m+n)+1-\alpha}}{\binom{2\alpha+1}{\alpha}}.$$

Conclusion

- Signature matrices of membranes generalize signature matrices of paths.
- We proved that there are no algebraic redundancies in the entries of signature matrices of membranes. We expect this to be true for higher level signature tensors as well.
- We showed that the id-signature of piecewise bilinear membranes can be computed efficiently, making it interesting for applications.
- Further study is necessary to understand which properties of a membrane are characterized by its id-signature.