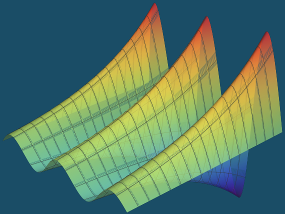

Signature matrices of membranes

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ACPMS Seminar
November 2024

Supported by TRR388 A04

Contents

The signature of a path

- A *path* is a continuous map $X : [0,1] \to V$ into some finite dimensional real vector space $V \cong \mathbb{R}^d$ such that all coordinate functions are piecewise continuously differentiable.
- Its *iterated integral signature*, introduced in the 1950s by Chen[1], is the linear form

$$\sigma(X) : T(V^*) \to \mathbb{R}$$
$$\alpha_1 \otimes \ldots \otimes \alpha_w \mapsto \int_{\Delta_w} \partial(\alpha_1 X)(t_1) \ldots \partial(\alpha_w X)(t_w) d\mathbf{t}$$

where $d\mathbf{t} = dt_1 \ldots dt_w$ and Δ_w is the w-simplex

$$\{(t_1, \ldots, t_w) \mid 0 \leq t_1 \leq \ldots \leq t_w \leq 1 \}$$

[1] K.-T. Chen (1954). "Iterated Integrals and Exponential Homomorphisms". In: *Proceedings of The London Mathematical Society*, pp. 502–512.

Properties of the path signature

- The signature $\sigma(X)$ **uniquely determines the path** X up to reparametrisation, translation and tree-like equivalence.
- There is a commutative product on $T(V^*)$ (called the shuffle product $⧢$) such that for every X the linear form $\sigma(X)$ defines an **algebra homomorphism** $(T(V^*), ⧢) \to \mathbb{R}$.
- Chen's identity: The signature is **compatible with concatenation** of paths:

$$\sigma(X \sqcup Y) = (\sigma(X) \otimes \sigma(Y)) \circ \Delta_{\otimes}$$

  Here $\Delta \colon T(V^*) \to T(V^*) \otimes T(V^*)$ is the *deconcatenation coproduct* of $T(V^*)$.
- Signatures of paths are central to the theory of rough paths. They are also commonly used in data science/machine learning as features for time series.

Signature matrices of paths

- The *second level signature* is the restriction $\sigma^{(2)}(X)$ of $\sigma(X)$ to $V^* \otimes V^*$. After choosing a basis of V, we can view this as a matrix $S(X)$.
- The sets

$$\{S(X) \mid X \textbf{ polynomial path in } \mathbb{R}^d \text{ of degree} \leq m\}$$

and

$$\{S(X) \mid X \textbf{ piecewise linear path in } \mathbb{R}^d \text{ with} \leq m \text{ segments}\}$$

agree[2].

- Their *Zariski closure* $\mathcal{P}^{\text{path}}_{d,m} \subseteq \mathbb{R}^{d \times d}$, that is, the zero set of all polynomials in the matrix entries vanishing on this set, is the *variety of path signature matrices* studied by Améndola, Friz, and Sturmfels in loc. cit..

[2] C. Améndola, P. Friz, and B. Sturmfels (2019). "Varieties of signature tensors". In: *Forum of Mathematics, Sigma* 7.

Signature matrices of paths

- For $m \leq d$, $\mathcal{P}^{\text{path}}_{d,m}$ agrees with the determinantal variety of matrices A such that A^{sym} has rank 1 and

$$\operatorname{rk}\begin{pmatrix} A^{\text{sym}} & A^{\text{sk}} \end{pmatrix} \leq m.$$

  Its dimension is $md - \binom{m}{2}$.
- For $m \geq d$, $\mathcal{P}^{\text{path}}_{d,m}$ stabilises and is called the *universal variety*. Its dimension is $\binom{d+1}{2}$ and it is cut out by the relation $\operatorname{rk} A^{\text{sym}} = 1$.

$\rightarrow$ Question: does this generalize to membranes?

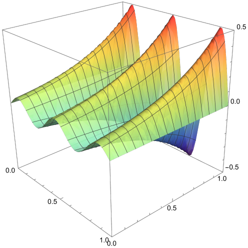
Signature matrices of membranes

The signature of a membrane

- A membrane is a continuous map $X : [0,1]^2 \to V$ into some f.d. real vector space V such that αX is a piecewise[3] continuously differentiable function for all $\alpha \in V^*$.

Example

$[0,1]^2 \to \mathbb{R}^3, (s,t) \mapsto (s, t, \frac{1}{2e^2} \sin(5\pi s) \exp(2t))$



[3] here: on a rectangular partition of $[0,1]^2$

The signature of a membrane

- There are different generalizations of the iterated-integral signature from paths to membranes.[4][5]
- We studied the most "naive" such generalization, which was introduced as the **id-signature** by Diehl, Ebrahimi-Fard, Harang, and Tindel in loc. cit..
- The main advantage of this approach is that the signature is still just a linear form on the same tensor algebra as for paths, simplifying calculations.

[4] J. Diehl, K. Ebrahimi-Fard, F. Harang, and S. Tindel (2024). *On the signature of an image*. arXiv: 2403.00130 [math.CA].

[5] C. Giusti, D. Lee, V. Nanda, and H. Oberhauser (2025). "A topological approach to mapping space signatures". In: *Advances in Applied Mathematics* 163, p. 102787. ISSN: 0196-8858.

Definition (The (id-)signature[6])

Let V be a f.d. real vector space and $X : [0,1]^2 \to V$ a membrane. Let $\partial_{12} := \frac{\partial^2}{\partial s \partial t}$.

Then the **(id-)signature** of X is the linear form

$$\sigma(X) : T(V^*) \to \mathbb{R},$$

$$\alpha_1 \otimes \ldots \otimes \alpha_w \mapsto \int_{\Delta_w^p} \partial_{12}(\alpha_1 X)(s_1, t_1) \ldots \partial_{12}(\alpha_w X)(s_w, t_w) \, \mathrm{d}\mathbf{s} \mathrm{d}\mathbf{t}$$

where $\mathrm{d}\mathbf{t} = \mathrm{d}t_1 \ldots \mathrm{d}t_w$, $\mathrm{d}\mathbf{s} = \mathrm{d}s_1 \ldots \mathrm{d}s_w$ and $\Delta_w^2 = \Delta_w \times \Delta_w$.

The *second level signature* is the restriction of $\sigma(X)$ to $V^* \otimes V^*$, and after choosing a basis of V we can view this as a matrix $S(X)$ which we call the **signature matrix** of X.

[6] J. Diehl, K. Ebrahimi-Fard, F. Harang, and S. Tindel (2024). *On the signature of an image*. arXiv: 2403.00130 [math.CA].

Some easy observations

Some features of this definition are:

- **Equivariance:** If $X\colon [0,1]^2 \to V$ is a membrane and $A\colon V \to W$ is a linear map then

$$\sigma(A \circ X) = \sigma(X) \circ T(A^*)$$

  In particular, $S(A \circ X) = AS(X)A^\top$.
- If two membranes only differ by a function in s or a function in t, they have the same signature. Thus, for any membrane X there is a *reduced membrane* X^{red} with $\sigma(X) = \sigma(X^{red})$ and $X^{red}([0,1] \times \{0\} \cup \{0\} \times [0,1]) = 0$.

An example

- Consider the bilinear membrane $X : [0,1]^2 \to \mathbb{R}^d, (s,t) \mapsto st \cdot u$ for $u \in \mathbb{R}^d$.
- The integrand in the integral associated to $(e_{i_1})^* \otimes \ldots \otimes (e_{i_k})^*$ under the signature is the constant

$$u_{i_1} \ldots u_{i_k}.$$

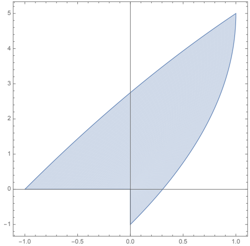
- This is integrated over Δ_k^2. The simplex Δ_k has volume $\frac{1}{k!}$ and thus

$$\langle \sigma(X), (e_{i_1})^* \otimes \ldots \otimes (e_{i_k})^* \rangle = \frac{1}{(k!)^2} u_{i_1} \ldots u_{i_k}$$

- In particular,

$$S(X) = \frac{1}{4} u \cdot u^\top$$

E.g., consider the membrane
$[0,1]^2 \to \mathbb{R}^2, (s,t) \mapsto (2st - s^2, 6st - t^2)$.



Its reduced membrane is $[0,1]^2 \to \mathbb{R}^d, (s,t) \mapsto st \cdot \begin{pmatrix} 2 \\ 6 \end{pmatrix}$. Thus, its signature matrix is

$$\frac{1}{4}\begin{pmatrix} 2\cdot 2 & 2\cdot 6 \\ 6\cdot 2 & 6\cdot 6 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$$

A property which is special to the id-signature is the following:

Lemma (Product membranes)

Let $X : [0,1] \to V$ *and* $Y : [0,1] \to W$ *be two paths and* V, W *finite dimensional. Let* $X \boxtimes Y$ *denote the induced membrane*

$$[0,1]^2 \to V \times W \to V \otimes W$$

Then, choosing a suitable basis for $V \otimes W$*, we have*

$$S(X \boxtimes Y) = S(X) \otimes S(Y)$$

where $\otimes$ *denotes the Kronecker product.*

Remark: in fact, the whole signature can be factored into the signatures of the two paths.

Example: a product membrane

Let $m : [0,1] \to \mathbb{R}^2, t \mapsto (t, t^2)$ be the moment path. Its signature matrix is

$$\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

The product membrane $m \boxtimes m$ can be viewed as the map $[0,1]^2 \to \mathbb{R}^{2\times 2}$ mapping (s,t) to (st, st^2, s^2t, s^2t^2). We obtain its signature matrix:

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{4}{9} \\ \frac{1}{6} & \frac{1}{4} & \frac{2}{9} & \frac{1}{3} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{6} & \frac{1}{6} & \frac{1}{4} \end{pmatrix}$$

It follows that e.g. $[0,1]^2 \to \mathbb{R}, (s,t) \mapsto (st + s^2t^2)$ has signature matrix $(\frac{1}{4} + \frac{4}{9} + \frac{1}{9} + \frac{1}{4}) = (\frac{19}{18})$.

Downsides of the id-signature

- There is a large class of membranes with vanishing signature; this includes for example the 2-sphere, parametrized spherically.
- An important property of the path signature is that it is an algebra homomorphism once $T(V^*)$ is equipped with the shuffle product. This is not true for membranes. In fact, in general the entries of signature matrices satisfy no algebraic relations at all ($\rightarrow$ later in this talk).

Signature matrices of membranes

1. The signature of a path
2. The signature of a membrane
3. $\mathcal{P}_{d,m,n}$: a variety of signature matrices
4. The dimension of $\mathcal{P}_{d,m,n}$
5. Open questions and outlook

A tale of two membranes

We focus on two classes of membranes:

- A **polynomial membrane** of order (m, n) is a membrane $X : [0,1]^2 \to \mathbb{R}^d$ that is polynomial in both s and t with bidegree $\leq (m, n)$.
- A **piecewise bilinear membrane** of order (m, n) is a membrane X such that there are $s_0 = 0 \leq s_1 \leq \cdots \leq s_m = 1$ and $t_0 = 0 \leq t_1 \leq \cdots \leq t_n = 1$ with X biaffine on all squares $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$.

A polynomial membrane

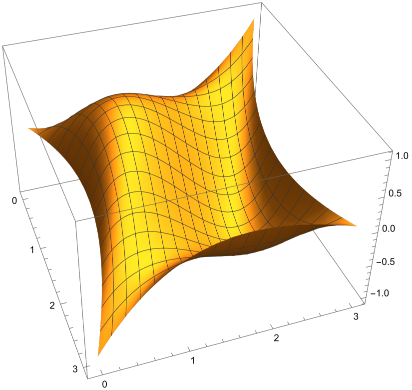


Figure: A Bezier surface with 4×4 control points is a polynomial membrane of order $(3, 3)$.

A piecewise bilinear membrane

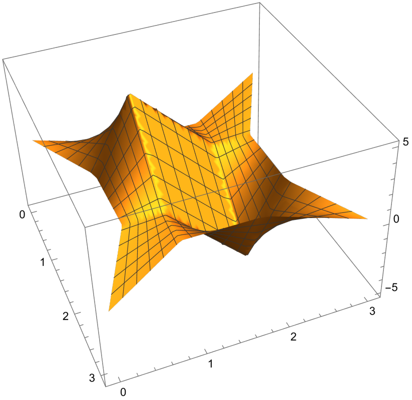


Figure: A piecewise bilinear membrane with 4×4 control points, that is, of order $(3, 3)$.

Another Bezier surface

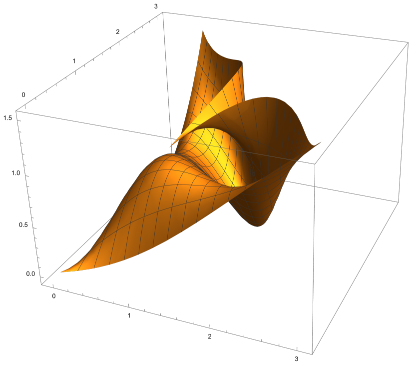


Figure: A membrane does not have to be the graph of a function.

Sets of signature matrices

Proposition (L., Schmitz '24)

The sets of signature matrices

$$\{S(X) \mid X \textbf{ polynomial membrane } \textit{in } \mathbb{R}^d \textit{ of degree} \le (m,n)\}$$

and

$$\{S(X) \mid X \textbf{ piecewise bilinear membrane } \textit{in } \mathbb{R}^d \textit{ of order } (m,n)\}$$

agree.

- For the proof, we use the notion of *dictionaries*, introduced by Pfeffer, Seigal, and Sturmfels[7].

[7] M. Pfeffer, A. Seigal, and B. Sturmfels (2019). "Learning Paths from Signature Tensors". In: *SIAM Journal on Matrix Analysis and Applications*.

Dictionaries for membranes

- Given a set of membranes $\mathcal{S}$ in a space W, a membrane $D : [0,1]^2 \to V$ is called a dictionary for $\mathcal{S}$ if

$$\mathcal{S} = \{A \circ D \mid A \colon V \to W \text{ linear}\}$$

- If $\mathcal{S}$ admits a dictionary D, then by equivariance of the signature, the signature of any $X \in \mathcal{S}$ can be expressed in terms of the signature of D. Indeed, recall that

$$S(A \circ D) = AS(D)A^\top.$$

  $S(D)$ is called the *core matrix*.
- In particular, if $D \colon [0,1]^2 \to V$ and $E : [0,1]^2 \to V$ are dictionaries for sets $\mathcal{S}_D$ and $\mathcal{S}_E$ of membranes and there is some automorphism $A \colon V \to V$ with $S(E) = AS(D)A^\top$, then $\mathcal{S}_D = \mathcal{S}_E$.

Dictionaries for membranes

- Key observation: there are dictionaries for polynomial membranes of order (m, n) as well as dictionaries for piecewise bilinear membranes of order (m, n), and they are obtained as products of dictionaries for polynomial paths or piecewise linear paths, respectively!
- For $k \in \mathbb{N}$, let Mom^k denote the path $t \mapsto (t, \ldots, t^k)$. The **moment membrane** of order (m, n) is defined as

$$\mathrm{Mom}^{m,n} := \mathrm{Mom}^m \boxtimes \mathrm{Mom}^n \colon [0,1]^2 \to \mathbb{R}^m \otimes \mathbb{R}^n$$

Proposition

$\mathrm{Mom}^{m,n}$ *is the dictionary for (reduced) polynomial membranes of order* (m, n). *In particular,*

$$S(\mathrm{Mom}^{m,n}) = S(\mathrm{Mom}^m) \otimes S(\mathrm{Mom}^n)$$

The axis membrane

- For $k \in \mathbb{N}$, let Axis^k denote the piecewise linear path with control points $(\sum_{i=1}^{j} e_i \mid 0 \leq j \leq k)$. The **axis membrane** of order (m, n) is defined as

$$\mathrm{Axis}^{m,n} := \mathrm{Axis}^m \boxtimes \mathrm{Axis}^n \colon [0,1]^2 \to \mathbb{R}^m \otimes \mathbb{R}^n$$

Proposition

$\mathrm{Axis}^{m,n}$ *is the dictionary for (reduced) piecewise bilinear membranes of order* (m, n)*. In particular,*

$$S(\mathrm{Axis}^{m,n}) = S(\mathrm{Axis}^m) \otimes S(\mathrm{Axis}^n)$$

Proposition

The set of signature matrices of polynomial membranes of order (m, n) agrees with the set of signature matrices of piecewise bilinear membranes of order (m, n).

Proof.

Améndola, Friz, and Sturmfels[8] show that there is an invertible matrix A_m with $A_m S(\mathrm{Axis}^m) A_m^\top = S(\mathrm{Mom}^m)$. Thus

$$(A_m \otimes A_n) S(\mathrm{Axis}^{m,n}) (A_m \otimes A_n)^\top = S(\mathrm{Mom}^{m,n}),$$

implying the claim. □

[8] C. Améndola, P. Friz, and B. Sturmfels (2019). "Varieties of signature tensors". In: *Forum of Mathematics, Sigma* 7.

The variety of signature matrices

- The set of signature matrices of polynomial membranes is the image of the morphism

$$\mathbb{R}^{d\times mn} \to \mathbb{R}^{d\times d}$$
$$A \mapsto A\, S(\mathrm{Axis}^{\mathrm{m,n}})\, A^\top .$$

  As this is a polynomial map (even homogenous of degree 2) the image will be a semi-algebraic subset of $\mathbb{R}^{d\times d}$.

- Tensoring with $\mathbb{C}$, we obtain a map

$$\phi : \mathbb{C}^{d\times mn} \to \mathbb{C}^{d\times d}$$

  We denote the Zariski closure of its image by $\mathcal{P}_{d,m,n}$. That is, $\mathcal{P}_{d,m,n}$ is the zero set of all polynomials in the matrix entries vanishing on the image. We call it the **variety of (membrane) signature matrices** (w.r.t. (d, m, n)).

The variety of signature matrices

Example

The variety $\mathcal{P}_{4,2,2}$ is the subvariety of $\mathbb{C}^{4\times 4}$ parametrised by the signature of piecewise linear membranes of order $(2,2)$ in $\mathbb{C}^4$. More precisely, it is the Zariski closure of the image of

$$\phi_{4,2,2} : \mathbb{C}^{4\times 4} \to \mathbb{C}^{4\times 4}$$

$$A \mapsto A \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} A^\top .$$

It has dimension 14 and agrees with the common zero set of two homogeneous polynomials in the matrix entries of $X \in \mathbb{C}^{4\times 4}$: the Pfaffian of the skew-symmetric matrix $X - X^\top$ and the quartic $\det(X) - \frac{1}{2}\det(X + X^\top)$.

The variety of signature matrices

Question: What is the dimension, degree and vanishing ideal of $\mathcal{P}_{d,m,n}$?

Signature matrices of membranes

The dimension of $\mathcal{P}_{d,m,n}$

- The dimension of a complex variety is just the dimension of its regular points as a complex manifold.
- By Chevalley's theorem, the image of

$$\begin{aligned} \phi : \mathbb{C}^{d \times mn} &\to \mathbb{C}^{d \times d} \\ A &\mapsto A \, S(\mathsf{Axis}^{\mathsf{m,n}}) \, A^\top. \end{aligned}$$

is a constructible set. Thus, it is dense in its Zariski closure.

The case $n = 1$ (or $m = 1$)

Proposition (L., Schmitz '24)

For $n = 1$, the variety $\mathcal{P}_{d,m,n}$ agrees with the variety $\mathcal{P}_{d,m}^{path}$ of signature matrices of piecewise linear paths with m segments in $\mathbb{R}^d$ (or polynomial paths of degree $\leq m$).

Proof.

We have

$$S(\text{Axis}^{m,1}) = S(\text{Axis}^m \otimes \text{Axis}^1) = S(\text{Axis}^m) \otimes S(\text{Axis}^1) = \frac{1}{2} S(\text{Axis}^m)$$

and $S(\text{Axis}^m)$ is the core tensor for piecewise linear paths with m segments. □

- In particular, if $m = 1$ or $n = 1$, the dimension of $\mathcal{P}_{d,m,n}$ will just be $\binom{d+1}{2}$.

The dimension for $mn \leq d$

Theorem (L., Schmitz '24)

For $mn \leq d$ the dimension of $\mathcal{P}_{d,m,n}$ is

- $dmn - \frac{1}{2}m^2n^2 + m^2(n-1) + (m-1)n^2 - \frac{7}{2}mn + 4(m+n) - 4$ *for even m, n.*
- $dmn - \frac{1}{2}m^2n^2 + m^2(n-1) + (m-1)n^2 - \frac{3}{2}mn + m + n$ *for even m, odd n.*
- $dmn - \frac{1}{2}m^2n^2 + m^2(n-1) + (m-1)n^2 - \frac{7}{2}mn + 3(m+n) - 2$ *for odd m, n.*

The case m odd, n even is omitted due to symmetry.

All polynomials agree with $md - \binom{m}{2}$ if $n = 1$, in accordance with the analogous result for paths[9].

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[9] C. Améndola, P. Friz, and B. Sturmfels (2019). "Varieties of signature tensors". In: *Forum of Mathematics, Sigma* 7.

The dimension for $mn \leq d$

Proof (sketch).

- After reducing to the case $d = mn$, one can use a theorem of De Terán and Dopico on the dimension of congruence orbits[10]. However, one needs to determine the *canonical form for congruence* of the matrix $S(\mathrm{Axis}^m \boxtimes \mathrm{Axis}^n)$.
- For this, one makes use of equivariance and the fact that $S(\mathrm{Axis}^m \boxtimes \mathrm{Axis}^n) = S(\mathrm{Axis}^m) \otimes S(\mathrm{Axis}^n)$ to reduce to the problem of finding the canonical form for congruence of $S(\mathrm{Axis}^m)$, which is easily calculated.

□

[10] F. D. Terán and F. M. Dopico (2011). "The solution of the equation $XA + AX^\top = 0$ and its application to the theory of orbits". In: *Linear Algebra and its Applications* 434.1, pp. 44–67.

The dimension for $m, n \gg d$

Contrary to the path case, $\mathcal{P}_{d,m,n}$ attains the full dimension d^2 for $m, n \gg d$. In fact:

Proposition (L., Schmitz '24)

We have $\mathcal{P}_{d,d,2} = \mathbb{R}^{d\times d}$, that is, the image of ϕ is Zariski-dense.

Proof.

Here, ϕ can be identified as the map $\mathbb{C}^{d\times 2d} \to \mathbb{C}^{d\times d}$ mapping two $d \times d$ matrices T and U to

$$TS(\mathrm{Axis}^{\mathrm{d}})T^\top + TS(\mathrm{Axis}^d)U^\top - US(\mathrm{Axis}^d)T^\top.$$

Its Jacobian at (Id, Id) is the linear map sending T and U to

$$2TS(\mathrm{Axis}^d) + S(\mathrm{Axis}^d)U^\top - US(\mathrm{Axis}^d).$$

The kernel of this map has dimension d^2 and the domain has dimension $2d^2$, so the Jacobian at (Id, Id) has full rank. □

Signature matrices of membranes

Mysterious zeroes

Computational data suggests that already $\mathcal{P}_{d,d-1,2}$ has full dimension. Something even stronger seems to be true:

Conjecture

The codimension of $\mathcal{P}_{d,m,n}$ *is* 0 *whenever* $m, n > 1$ *and* $m + n = d + 1$.

$$\begin{bmatrix} 56 & 49 & 43 & 38 & 34 & 31 & 29 & 28 \\ 49 & 34 & 21 & 12 & \mathbf{4} & \mathbf{2} & \mathbf{0} & 0 \\ 43 & 21 & \mathbf{12} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ 38 & 12 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 34 & \mathbf{4} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 31 & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 29 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure: The codimensions of $\mathcal{P}_{8,m,n}$ for $m, n = 1, \ldots, 8$.

Degree and vanishing ideal?

We do not understand the equations that cut out the varieties $\mathcal{P}_{d,m,n}$ in general, or even their degree. Using `NumericalImplicitization.m2` and `MultigradedImplicitization.m2` one can study special cases: we observed e.g. that the variety $\mathcal{P}_{5,3,2}$ has dimension 24 and is cut out by a single quintic with 999 terms.

GL-invariants in the tensor algebra

- As the signature of a membrane is equivariant under the diagonal GL_d action on $T(\mathbb{R}^d)$, GL-equivariants for paths will be GL-equivariants for membranes. In particular, one would still expect them to have a geometric interpretation.
- $\rightarrow$ What is the geometrical meaning of the "signed volume" for membranes?

Algebraic relations in the id-signature

We showed that the entries of signature matrices for membranes satisfy in general no algebraic relations.
In fact, we expect this to be true for the whole id-signature, not just the second level.

Conjecture

For any $d, k \in \mathbb{N}$ *and every polynomial* $p \neq 0$ *in the coordinates of* $(\mathbb{R}^d)^{\otimes k}$ *there is a membrane* $X : [0,1]^2 \to \mathbb{R}^d$ *such that* $p(\sigma^{(k)}(X)) \neq 0$*.*

arXiv:2409.11996