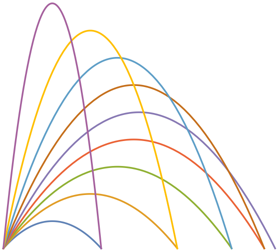

PathSignatures

Computing signature varieties in Macaulay2

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[1] joint with Carlos Améndola, Angelo El Saliby and Oriol Reig Fité

The signature of a path

Paths and tensors

Let V be a finite dimensional $\mathbb{R}$-vector space.

Definition

A path in V is a continuous, piecewise continuously differentiable map $X : [0,1] \to V$.

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Paths and tensors

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A path in V is a continuous, piecewise continuously differentiable map $X:[0,1]\to V$.

Theorem (Chen 1958[2])

There is a natural injection

$$\sigma : \{Paths\ in\ V\}/\sim \to T(V^\vee)^\vee$$

where $\sim$ identifies paths that are equal up to translation, reparametrization and tree-like excursions. Here,

$$T(V^\vee) := \bigoplus_{k\in\mathbb{N}_0} (V^\vee)^{\otimes k}.$$

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The signature of a path

Definition

Let X be a path in V. Then the signature $\sigma(X)$ of X is the linear form

$$\begin{aligned} T(V^\vee) &\to \mathbb{R} \\ \alpha_1 \otimes \ldots \otimes \alpha_k &\mapsto \int_{\Delta_k} (\alpha_1 \circ X)'(t_1) \ldots (\alpha_k \circ X)'(t_k) \, \mathrm{d}t_1 \ldots \mathrm{d}t_k \end{aligned}$$

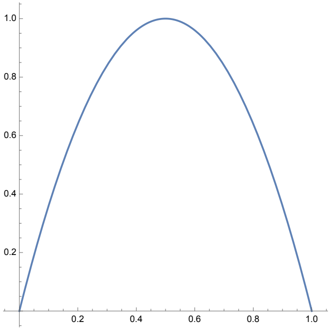
where Δ_k denotes the k-simplex

$$\{(t_1, \ldots, t_k) \mid 0 \le t_1 \le \ldots \le t_k \le 1\}.$$

The restriction of $\sigma(X)$ to $(V^\vee)^{\otimes k}$ can be viewed as an element of $V^{\otimes k}$ and it is called the *k-th level signature tensor* of X.

The signature of a path

Example



$$X : [0,1] \to \mathbb{R}^2,\ t \mapsto (t, 4t(1-t))$$

Let us compute $\sigma(X)(e_2^*)$ and $\sigma(X)(e_1^* \otimes e_2^*)$.

$$X:[0,1]\to\mathbb{R}^2,\ t\mapsto(t,4t(1-t))$$

$$\sigma(X)(e_2^*)=\int_{\Delta_1}(e_2^*\circ X)'(t)\mathrm{d}t=[4t(1-t)]_0^1=0$$

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$$\begin{aligned}\sigma(X)(e_1^* \otimes e_2^*) &= \int_{\Delta_2} (e_1^* \circ X)'(t_1)(e_2^* \circ X)'(t_2)\mathrm{d}t_1\mathrm{d}t_2 \\ &= \int_0^1 (e_2^* \circ X)'(t_2) \cdot [t_1]_0^{t_2}\mathrm{d}t_2 \\ &= \int_0^1 4t_2 - 8t_2^2 dt_2 = [2t_2^2 - \frac{8}{3}t_2^3]_0^1 = -\frac{2}{3}\end{aligned}$$

Encoding tensors in Macaulay2

- After choosing a basis $b_1, \ldots, b_d$ of V we can wlog. assume $V = \mathbb{R}^d$.

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- We can identify $T((\mathbb{R}^d)^\vee)$ resp. $T(\mathbb{R}^d)$ with the free associative algebra $\mathbb{R}\langle \mathtt{1}, \dots, \mathtt{d} \rangle$ over the alphabet $\mathtt{1}, \dots, \mathtt{d}$ via

$$e_1^* \otimes \dots \otimes e_d^* \mapsto \mathtt{i}_\mathtt{1} \dots \mathtt{i}_\mathtt{d}$$

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- We use this to encode tensors in Macaulay2 as non-commutative polynomials, provided by the package `NCAlgebra`. `PathSignatures` introduces some additional methods for working with these polynomials.

Encoding tensors in MACAULAY2

Example

```
i1 : needsPackage "PathSignatures";

i2 : A2 = wordAlgebra(2);

i3 : f = [1,2]_A2 - [2,1]_A2

o3 = -Lt Lt +Lt Lt
        2  1   1  2

o3 : A2

i4 : f // wordFormat

o4 = - [2, 1] + [1, 2]
```

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- `PathSignatures.m2` provides a type for (piecewise) polynomial paths. An instance is constructed by giving the coordinate functions.

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Example

```
i5 : QQ[t];
i6 : X = polyPath({t, 4 * t * (1-t)})
o6 = Path in 2-dimensional space with 1 polynomial segment:

                2
     {{t, - 4t  + 4t}}

o6 : Path
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- The signature of the path `X` is represented by the method `sig(X,·)` which takes a non-commutative polynomial as input.
- For an integer `k`, `sig(X,k)` returns the k-th level signature of X as a non-commutative polynomial.

The signature in Macaulay2

Example

```
i7 : sig(X, [2]_A2)

o7 = 0

i8 : sig(X, [1,2]_A2)

       2
o8 = - -
       3

i9 : sig(X, 2)

     2            2            1
o9 = - [2, 1] - - [1, 2] + - [1, 1]
     3            3            2
```

Interpretation of signatures

- The first level signature gives the *increment* of the path

---

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Interpretation of signatures

- The first level signature gives the *increment* of the path
- The second level signature encodes the *Levy area* of all projections of the path to two coordinates.

Example

```
i10 : sig(X,[1,2]_A2 - [2,1]_A2)

        4
o10 = - -
        3
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        3
```

- In general, signature values can be interpreted as "areas of areas"[3].

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Path signatures in theory and application

- Path signatures were originally studied by Chen[4] in the context of algebraic topology and differential geometry.

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- Later, they became central to the theory of rough paths in stochastic analysis.[5]
- Signatures are successfully used as a feature for paths in machine learning, for example in hand-writing recognition.[6]
- Iterated integrals of this form also appear in High Energy Physics and Number Theory (polylogarithms, MZVs, ...)

---

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$$\mathtt{12} \sqcup\!\sqcup \mathtt{34} = \mathtt{1234} + \mathtt{1324} + \mathtt{1342} + \mathtt{3124} + \mathtt{3142} + \mathtt{3412}$$

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- For any path X, the signature $\sigma(X)$ is an algebra homomorphism $(\mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d} \rangle, +, \sqcup\!\sqcup) \to (\mathbb{R}, +, \cdot)$.

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Example

$$12 \sqcup\!\sqcup 34 = 1234 + 1324 + 1342 + 3124 + 3142 + 3412$$

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Example

```
i11 : sh = [1,1]_A2 ** [2,2]_A2; sh // wordFormat

o12 = [2, 2, 1, 1] + [2, 1, 2, 1] + [2, 1, 1, 2] +
      ------------------------------------------------
      [1, 2, 2, 1] + [1, 2, 1, 2] + [1, 1, 2, 2]

i13 : sig(X, sh) == sig(X, [1,1]_A2) * sig(X, [2,2]_A2)
o13 = true
```

Concatenation of paths

- The subset of algebra homomorphisms in $(\mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d} \rangle)^\vee$ forms a Lie group $\mathcal{G}(\mathbb{R}^d)$ with the group multiplication given by folding:

$$f * g(\mathtt{i}_1 \ldots \mathtt{i}_k) = \sum_{j=0}^{k} f(\mathtt{i}_1 \ldots \mathtt{i}_j) g(\mathtt{i}_{j+1} \ldots \mathtt{i}_k).$$

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- If $X \bullet Y$ denotes the concatenation of paths X and Y then $\sigma(X \bullet Y) = \sigma(X) * \sigma(Y)$ (*Chen's identity*).
- `PathSignatures` uses this to compute the signature of *piecewise polynomial paths*.

Concatenation of paths

Example

```
i1 : needsPackage "PathSignatures";
i2 : S = QQ[a_(1,1)..a_(3,2)];
i3 : X = linPath({a_(1,1),a_(1,2)}) ** linPath({a_(2,1),a_(2,2)}) ** linPath({a_(3,1),a_(3,2)})
o3 = Path in 2-dimensional space with 3 polynomial segments:

     {{a   t, a   t}, {a   t, a   t}, {a   t, a   t}}
        1,1    1,2      2,1    2,2      3,1    3,2

o3 : Path
i4 : A2 = wordAlgebra(2);
i5 : sig(X,[1,2]_A2 - [2,1]_A2)

o5 = - a   a    + a   a    - a   a    - a   a    + a   a    + a   a
        1,2 2,1    1,1 2,2    1,2 3,1    2,2 3,1    1,1 3,2    2,1 3,2

o5 : S
```

Varieties of signature tensors

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- Goal: understand connections between paths and the signature tensors that represent them.

[7] C. Améndola, P. Friz, and B. Sturmfels. "Varieties of signature tensors". In: *Forum of Mathematics, Sigma* 7 (2019).

Varieties of signature tensors

- Goal: understand connections between paths and the signature tensors that represent them.
- From an algebra-geometric viewpoint, this leads to the study of *signature varieties* for families of paths.[7] They are obtained as the Zariski closure of the set of (k-th level) signature tensors of paths in the family.

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- Main point: in the case where the parametrization is polynomial, determining the signature tensor variety is an *implicitization problem*.

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- Main point: in the case where the parametrization is polynomial, determining the signature tensor variety is an *implicitization problem*.
- $\rightarrow$ Macaulay2, `NumericalImplicitization`, `MultigradedImplicitization` ...

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Varieties of signature tensors

Example

Consider degree 2 polynomial paths in $\mathbb{R}^2$. This family is parametrized by matrices $A \in \mathbb{R}^{2\times 2}$:

$$X_A := (a_{11}t + a_{12}t^2, a_{21}t + a_{22}t^2)$$

It is immediate from the definition that the entries of $\sigma^{(k)}(X_A)$ are homogeneous degree k polynomials in the entries of A. For example, $\sigma^{(2)}(X_A)$ is given by

$$\begin{pmatrix} \frac{1}{2}a_{11}^2 + a_{11}a_{12} + \frac{1}{2}a_{12}^2 & \frac{1}{2}a_{11}a_{21} + \frac{1}{3}a_{12}a_{21} + \frac{2}{3}a_{11}a_{22} + \frac{1}{2}a_{12}a_{22} \\ \frac{1}{2}a_{11}a_{21} + \frac{2}{3}a_{12}a_{21} + \frac{1}{3}a_{11}a_{22} + \frac{1}{2}a_{12}a_{22} & \frac{1}{2}a_{21}^2 + a_{21}a_{22} + \frac{1}{2}a_{22}^2 \end{pmatrix}$$

The associated signature matrix variety in $\mathbb{R}^{2\times 2}$ is cut out by the equation $X_{11}X_{22} - (\frac{1}{2}(X_{12} + X_{21}))^2$, the determinant of the symmetric part of the matrix.

Using PathSignatures

Example

```
i1 : needsPackage "PathSignatures";
i2 : S = QQ[a_(1,1)..a_(2,2)];
i3 : R = S[t];
i4 : X = polyPath({a_(1,1) * t + a_(1,2) * t^2,
                   a_(2,1) * t + a_(2,2) * t^2})
o4 = Path in 2-dimensional space with 1 polynomial
     segment:

           2              2
   {{a   t  + a   t, a   t  + a   t}}
      1,2      1,1    2,2      2,1

o4 : Path
```

Using PathSignatures

Example

```
i5 : A2 = wordAlgebra(2, CoefficientRing => S);

i6 : sig(X,[1,2]_A2)

     1           1           2           1
o6 = -a   a    + -a   a    + -a   a    + -a   a
     2 1,1 2,1   3 1,2 2,1   3 1,1 2,2   2 1,2 2,2

o6 : S
```

Using PathSignatures

Example

```
i7 : T = sig(X,2);

i8 : m = tensorParametrization(T);

o8 : RingMap S <-- QQ[b      , b      , b      , b      ]
                       [1, 2]   [2, 2]   [2, 1]   [1, 1]

i9 : kernel m

             2                         2
o9 = ideal(b       + 2b      b       + b
            [1, 2]      [1, 2] [2, 1]     [2, 1]

                                             - 4b      b      )
                                                 [2, 2] [1, 1]
```

Using PathSignatures

Example

```
i10 : T = sig(X,4);

i11 : m = tensorParametrization(T);

o11 : RingMap S <-- QQ[...]

i12 : I = kernel m;

o12 : Ideal of QQ[...]

i13 : dim I

o13 = 4

i14 : degree I

o14 = 24
```

Using PathSignatures

Example

```
i15 : betti mingens I

             0  1
o15 = total: 1 56
          0: 1  1
          1: . 55

o15 : BettiTally
```

This reproduces the data from Table 2 in "Varieties of signature tensors"[8].

[8] C. Améndola, P. Friz, and B. Sturmfels. "Varieties of signature tensors". In: *Forum of Mathematics, Sigma* 7 (2019).

arXiv:2506.01429