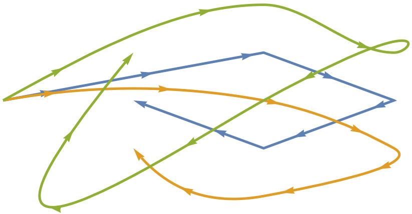

Signature Varieties of Splines

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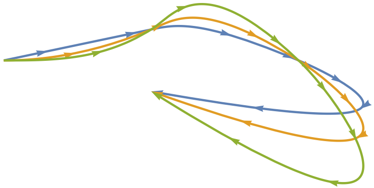
Piecewise polynomial paths

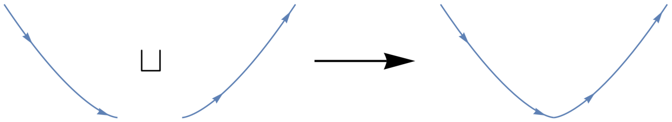
- A **polynomial path** is a map $X : [0, 1] \to \mathbb{R}^d$ whose coordinate functions are given by polynomials. A **piecewise polynomial path** is a continuous map $X : [0, 1] \to \mathbb{R}^d$ such that there exist $0 = t_0 < t_1 < \ldots < t_\ell = 1$ with each $X|_{[t_i, t_{i+1}]}$ a polynomial path.
- A **parametric spline** is a piecewise polynomial path satisfying certain regularity constraints.
- Splines play an important role in applications, e.g. interpolation of discrete data and modelling (Bézier curves, ...).



Concatenation of paths

- If $X^{[1]}, \ldots, X^{[\ell]}$ are polynomial paths in $\mathbb{R}^d$, their **concatenation** is the piecewise polynomial path

$$(X^{[1]} \sqcup \ldots \sqcup X^{[\ell]})(t) := \begin{cases} X^{[1]}(\ell t) & t \in [0, \frac{1}{\ell}] \\ X^{[1]}(1) + X^{[2]}(\ell(t - \frac{1}{\ell})) & t \in [\frac{1}{\ell}, \frac{2}{\ell}] \\ \ldots & \ldots \\ \sum_{i=1}^{\ell-1} X^{[i]}(1) + X^{[\ell]}(\ell(t - \frac{\ell-1}{t})) & t \in [\frac{\ell-1}{\ell}, 1] \end{cases}$$

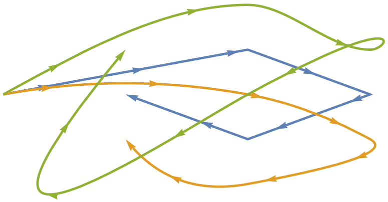


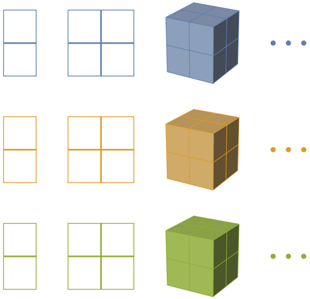
Concatenation of paths

- Up to reparametrization, every piecewise polynomial path is of this form.

The path signature

(Piecewise polynomial) paths $\xrightarrow[\int\int\int\cdots]{\text{Path signature}}$ Collections of tensors





The signature of a path

Definition (Chen '58)
Let X be a path in $\mathbb{R}^d$. Then the k-truncated signature $\sigma^{\leq k}(X)$ of X is the collection of tensors

$$(\sigma^{(1)}(X), \sigma^{(2)}(X), \ldots, \sigma^{(k)}(X)) \in \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} \oplus \ldots \oplus (\mathbb{R}^d)^{\otimes k}$$

where the i-th *signature tensor* $\sigma^{(i)}(X) \in (\mathbb{R}^d)^{\otimes i}$ is defined by

$$\sigma^{(i)}(X)_{w_1 \ldots w_i} := \int_{\Delta_i} \dot{X}_{w_1}(t_1) \ldots \dot{X}_{w_i}(t_i)\, \mathrm{d}t_1 \ldots \mathrm{d}t_i.$$

Here Δ_i denotes the i-simplex

$$\{(t_1, \ldots, t_\ell) \mid 0 \leq t_1 \leq \ldots \leq t_\ell \leq 1\}$$

and $X_1, \ldots, X_d$ denote the coordinate functions of X.

The path signature

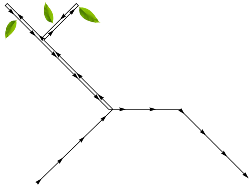
- Introduced by Kuo-Tsai Chen in the 50s[1] in context of differential geometry and topology.
- Later became foundational in the theory of rough paths in stochastic analysis[2].
- Applications in quantitative finance, machine learning, ...

[1] K.-T. Chen. **"Integration of Paths–A Faithful Representation of Paths by Noncommutative Formal Power Series".** In: **Transactions of the American Mathematical Society** 89.2 (1958), pp. 395–407.
[2] T. J. Lyons. **"Differential equations driven by rough signals.".** eng. In: **Revista Matemática Iberoamericana** 14.2 (1998), pp. 215–310.

Recovery and redundancy

Theorem (Chen '58)

A path X is **uniquely characterized** *by the collection of* **all** *tensors $\sigma^{(i)}(X), i \in \mathbb{N}$ up to translation, reparametrization and tree-like equivalence.*



Removing the 'branches' yields a tree-like equivalent path.

- $\rightarrow$ given $\sigma^{\leq k}(X)$ for a path X in $\mathbb{R}^d$, how well can we **recover** X if we assume that X is from a given family of paths $\mathcal{X}$?
- $\rightarrow$ given a family of paths $\mathcal{X}$, what are the **algebraic relations** among the entries of $\sigma^{\leq k}(X)$ for generic $X \in \mathcal{X}$?

We study these question for **families of splines**.

Families of splines

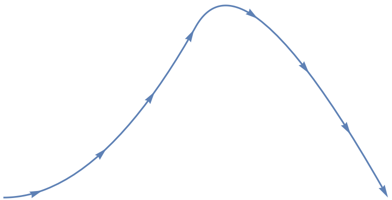
- Let $d, \ell \in \mathbb{N}$, $m = (m_1, \ldots, m_\ell) \in \mathbb{N}^\ell$ and $r \in \mathbb{N}_0$. Let $\mathcal{X}_{d,m,r}$ be the family of piecewise polynomial paths of the form

$$X^{[1]} \sqcup \ldots \sqcup X^{[\ell]}$$

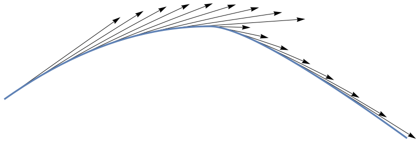
where $X^{[i]}$ is a polynomial path of degree m_i such that

$$\frac{\mathrm{d}^s X^{[i+1]}}{\mathrm{d}t^s}(0) = \rho_{i,s} \frac{\mathrm{d}^s X^{[i]}}{\mathrm{d}t^s}(1)$$

for some $\rho_{i,s} > 0$ and all $i < \ell$ and $1 \leq s \leq r$.



A path in $\mathcal{X}_{2,(2,2),1}$



A 1-regular path with discontinuous parametrization

Signature varieties of splines

$$\mathcal{X}_{d,m,r}$$

ambient dimension *degrees of pieces* *regularity*

We call an element of $\mathcal{X}_{d,m,r}$ a *geometric* m-spline in $\mathbb{R}^d$ of regularity r.

Definition
Let $d, r, \ell \in \mathbb{N}$ and $m \in \mathbb{N}^\ell$. The **(k-truncated) signature variety** of geometric m-splines in $\mathbb{R}^d$ of regularity r is

$$\mathcal{S}^r_{d,\leq k,m} := \overline{\{\sigma^{\leq k}(X) \mid X \in \mathcal{X}_{d,m,r}\}} \subseteq \bigoplus_{i \leq k} (\mathbb{C}^d)^{\otimes i}.$$

Signature varieties of splines

$$\mathcal{S}^r_{d,\leq k,m} := \overline{\{\sigma^{\leq k}(X) \mid X \in \mathcal{X}_{d,m,r}\}} \subseteq \bigoplus_{i \leq k} (\mathbb{C}^d)^{\otimes i}.$$

- We can **parametrize** the paths in $\mathcal{X}_{d,m,r}$ by parameters $\mathbf{a} \in \mathbb{R}^{d \times \kappa}$ and $\rho \in \mathbb{R}^{\ell \times r}$, where $\kappa = \sum_{i=1}^{\ell} m_i - (\ell - 1)r$.
- If $X_{\mathbf{a},\rho}$ is the path corresponding to parameters $\mathbf{a}$ and ρ, then the signature $\sigma^{\leq k}(X_{\mathbf{a},\rho})$ is **polynomial** in $\mathbf{a}$ and ρ.
- In particular, $\mathcal{S}^r_{d,\leq k,m}$ is the solution to an **implicitization problem**.
- The varieties $\mathcal{S}^0_{d,\leq k,(M)}$ (for $M \in \mathbb{N}$) and $\mathcal{S}^0_{d,\leq k,(1,\ldots,1)}$ are the signature varieties of polynomial and piecewise linear paths, respectively; studied previously in Améndola, Friz, and Sturmfels [AFS19].

The free k-step nilpotent Lie group

$$\begin{tikzcd} \mathbb{C}^{d\times\kappa} \oplus \mathbb{C}^{\ell\times r} \arrow[rr, "\sigma^{\leq k}(X_{\mathbf{a},\rho})"] \arrow[drr] & & \bigoplus_{i\leq k}(\mathbb{C}^d)^{\otimes i} \\ & & \mathcal{G}^{\leq k}(\mathbb{C}^d) \arrow[u] \end{tikzcd}$$

$$\kappa = \sum_{i=1}^{\ell} m_i - (\ell - 1)r$$

- The signature always factors through a subvariety $\mathcal{G}^{\leq k}(\mathbb{C}^d)$ of $\bigoplus_{i\leq k}(\mathbb{C}^d)^{\otimes i}$, the **free k-step nilpotent Lie group**.
- It is cut out by **shuffle relations**, which are general relations of iterated integrals following from integration by parts.

$$\Longrightarrow \quad \mathcal{S}^r_{d,\leq k,m} \subseteq \mathcal{G}^{\leq k}(\mathbb{C}^d)$$

An example

A path in $\mathcal{X}_{2,(2,2),1}$ is of the form

$$(a_{1,1}t + a_{1,2}t^2, a_{2,1}t + a_{2,2}t^2) \sqcup (\rho_{1,1}(a_{1,1} + 2a_{1,2})t + a_{3,2}t^2, \rho_{1,1}(a_{2,1} + 2a_{2,2})t + a_{4,2}t^2).$$

Write $X_{\mathbf{a},\rho}$ for this path, then[3]

$$\sigma^{\leq 2}(X_{\mathbf{a},\rho}) = \begin{pmatrix} a_{1,1}\rho_{1,1} + 2\,a_{1,2}\rho_{1,1} + a_{1,1} + a_{1,2} + a_{3,2} \\ a_{2,1}\rho_{1,1} + 2\,a_{2,2}\rho_{1,1} + a_{2,1} + a_{2,2} + a_{4,2} \end{pmatrix} \oplus \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

$$\begin{aligned}
x &= \frac{1}{2}\left[(a_{1,1} + 2a_{1,2})\,\rho_{1,1} + (a_{1,1} + a_{1,2} + a_{3,2})\right]^2 \\
y &= \frac{1}{2}(a_{1,1} + 2a_{1,2})(a_{2,1} + 2a_{2,2})\,\rho_{1,1}^2 + \left[(a_{2,1} + 2a_{2,2})(a_{1,1} + a_{1,2} + \tfrac{1}{3}a_{3,2}) + \tfrac{2}{3}(a_{1,1} + 2a_{1,2})a_{4,2}\right]\rho_{1,1} \\
&\quad + (a_{1,1} + a_{1,2} + \tfrac{1}{2}a_{3,2})a_{4,2} + \tfrac{1}{2}a_{2,1}(a_{1,1} + \tfrac{2}{3}a_{1,2}) + \tfrac{1}{2}a_{2,2}(\tfrac{4}{3}a_{1,1} + a_{1,2}) \\
z &= \frac{1}{2}(a_{1,1} + 2a_{1,2})(a_{2,1} + 2a_{2,2})\,\rho_{1,1}^2 + \left[(a_{2,1} + 2a_{2,2})(a_{1,1} + a_{1,2} + \tfrac{2}{3}a_{3,2}) + \tfrac{1}{3}(a_{1,1} + 2a_{1,2})a_{4,2}\right]\rho_{1,1} \\
&\quad + (a_{2,1} + a_{2,2} + \tfrac{1}{2}a_{4,2})a_{3,2} + \tfrac{1}{2}a_{2,1}(a_{1,1} + \tfrac{2}{3}a_{1,2}) + \tfrac{1}{2}a_{2,2}(\tfrac{4}{3}a_{1,1} + a_{1,2}) \\
w &= \frac{1}{2}\left[(a_{2,1} + 2a_{2,2})\,\rho_{1,1} + (a_{2,1} + a_{2,2} + a_{4,2})\right]^2
\end{aligned}$$

[3] F. Lotter, O. Reig, A. E. Saliby, and C. Amendola. **PathSignatures: working with algebraic path signatures. Version 1.0.** A **Macaulay2** package.

An example

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \oplus \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \in \mathbb{C}^2 \oplus (\mathbb{C}^2)^{\otimes 2}$$

The following polynomials vanish on the image of $\sigma^{\leq 2}(X_{\mathbf{a},\rho})$:

$$\begin{aligned}
T_2^2 - 2T_{2,2} \\
T_1 T_2 - T_{2,1} - T_{1,2} \\
T_1^2 - 2T_{1,1} \\
2T_{2,2}T_1 - T_{2,1}T_2 - T_{1,2}T_2 \\
T_{2,1}T_1 + T_{1,2}T_1 - 2T_2T_{1,1} \\
T_{2,1}^2 + 2T_{2,1}T_{1,2} + T_{1,2}^2 - 4T_{2,2}T_{1,1}
\end{aligned}$$

They cut out the three-dimensional variety $\mathcal{S}^1_{2,\leq 2,(2,2)}$. In fact, $\mathcal{S}^1_{2,\leq 2,(2,2)} = \mathcal{G}^{\leq 2}(\mathbb{C}^2)$.

Inclusions and isomorphisms

regularity

$$\mathcal{S}^r_{d,\leq k,m}$$

ambient dimension   *truncation level*   *degrees of pieces*

- We have inclusions $\mathcal{S}^r_{d,\leq k,m} \subseteq \mathcal{S}^{r'}_{d,\leq k,m}$ for $r' \leq r$. On the other hand, we have inclusions of the type $\mathcal{S}^r_{d,\leq k,(2,2)} \subseteq \mathcal{S}^r_{d,\leq k,(3,3)} \subseteq \mathcal{S}^r_{d,\leq k,(3,1,2,\ldots,3)}, \cdots$
- There is some N with $\mathcal{S}^r_{d,\leq k,m} = \mathcal{G}^{\leq k}(\mathbb{C}^d)$ once $|m| = m_1 + \ldots + m_\ell \geq N$.
- For $m = (m_1, \ldots, m_\ell)$, set $R(m) := (m_\ell, \ldots, m_1)$. Then

$$\mathcal{S}^r_{d,\leq k,m} \cong \mathcal{S}^r_{d,\leq k,R(m)}$$

under a linear automorphism of $\bigoplus_{i\leq k}(\mathbb{C}^d)^{\otimes i}$.

Matrix varieties

Write $\mathcal{S}^r_{d,2,m}$ for the image (closure) of $\mathcal{S}^r_{d,\leq 2,m}$ under $\mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes 2} \to (\mathbb{C}^d)^{\otimes 2}$. Then $\mathcal{S}^0_{d,(M),2}$ for $M \in \mathbb{N}$ are the matrix varieties for polynomial paths from Améndola, Friz, and Sturmfels 2019 [AFS19].

Proposition (Améndola, L., Schmitz '26)

For all $d, M \in \mathbb{N}$ and $m = (m_1, \ldots, m_\ell)$ with $m_1 + \ldots + m_\ell = M$, we have

$$\mathcal{S}^r_{d,2,m} = \mathcal{S}^0_{d,2,(M-(\ell-1)r)}.$$

It has dimension $dM - (\ell-1)dr - \binom{M-(\ell-1)r}{2}$.

Theorem (Améndola, L., Schmitz '26)

If M is odd, then

$$\deg(\mathcal{S}^0_{d,2,M}) = 2^{M-1} \prod_{n=0}^{d-M-1} \frac{\binom{d+n}{d-M-n}}{\binom{2n+1}{n}}.$$

Higher tensor varieties

We use symbolic[4] and numerical[5] tools to compute dimension and degree of higher tensor varieties. **Observation:** in most cases, the dimension is the minimum of $\dim \mathcal{G}^{\leq k}(\mathbb{C}^d)$ and the number of parameters.

$m \setminus r$	0	1	2
$(2,2)$	8	7	<u>6</u>
$(2,1,1)$	8	5	-
$(2,1)$	<u>6</u>	5	-
$(1,1,1)$	6	2	-

(a) Dimension

$m \setminus r$	0	1	2
$(2,2)$	53	?	<u>552</u>
$(2,1,1)$	53	414	-
$(2,1)$	<u>552</u>	414	-
$(1,1,1)$	276	16	-

(b) Degree (in $\bigoplus_{i\leq 4}(\mathbb{C}^d)^{\otimes i}$)

Table: Dimension and degree for $\mathcal{S}^r_{2,\leq 4,m}$.

Exception: $\dim \mathcal{S}^2_{3,\leq 3,(2,2)} = 7$, but $\dim \mathcal{G}^{\leq 3}(\mathbb{C}^3) = 14$ and there are 8 parameters.

[4] Macaulay2, `msolve` [BES21]

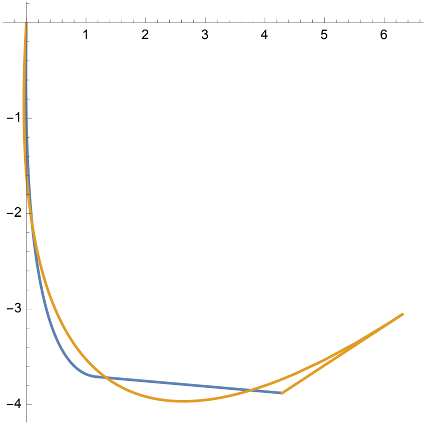
[5] `NumericalImplicitization.m2` [CK19], `HomotopyContinuation.jl` [BT18]

Fibers and path recovery

- Let us consider cases where the parametrizing signature map is **generically finite-to-one**.
- Then we call the number of points in a generic fiber of the signature map the **path recovery degree** and denote it $\mathrm{PRdeg}(\mathcal{S}^r_{d,\leq k,m})$.
- The **real solutions** correspond to paths; among these, the solutions with $\rho > 0$ correspond to splines. Solutions with $\rho < 0$ are paths with 'cusps'.

Path recovery for $d = 2, k = 3, m = (2, 1)$

$$\mathrm{PRdeg}(\mathcal{S}^1_{2,\leq 3,(2,1)}) = 2$$



The spline solution is unique if it exists.

Path recovery degrees

m	r	$\mathrm{PRdeg}(\mathcal{S}^r_{2,\leq 4,m})$
$(1,1,1,1)$	0	4
$(1,2,1)$	0	18
$(2,1,1)$	0	14
$(2,2)$	0	10
$(3,1)$	0	40
(4)	0	48
$(2,2,1)$ $(2,1,2)$	1	32
$(1,3,1)$	1	96
$(3,2)$	2	116

Table: Recovery degrees for geometric m-splines in the plane from the 4-truncated signature.

Path recovery degrees

m	r	$\mathrm{PRdeg}(\mathcal{S}^r_{3,\leq 3,m})$
$(3,2,1)$	1	144
$(3,1,2)$	1	84
$(2,2,2)$	1	90
$(4,2)$	2	312
$(3,3)$	2	168

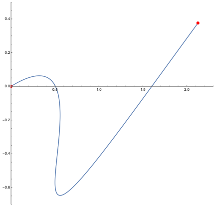
Table: Recovery degrees for geometric m-splines in 3-space from the 3-truncated signature.

Here, our computations only terminated with `msolve`[6]!

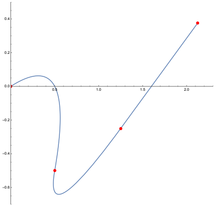
[6] J. Berthomieu, C. Eder, and M. Safey El Din. **"msolve: A Library for Solving Polynomial Systems".** In: **Proceedings of the 2021 International Symposium on Symbolic and Algebraic Computation**. ISSAC '21. Virtual Event, Russian Federation: Association for Computing Machinery, 2021, pp. 51–58.

4 paths with the same 4-truncated signature

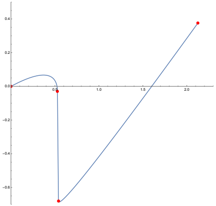
Using `HomotopyContinuation.jl`[7], we can reconstruct *m*-splines with identical 4-truncated signatures but different *m*:



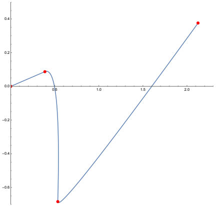
$m = (4)$



$m = (2, 2, 1)$



$m = (2, 1, 2)$



$m = (1, 2, 2)$

Splines of regularity 1 for different *m*, with the same signature up to level 4. Control points are marked in red.

[7] P. Breiding and S. Timme. **"HomotopyContinuation.jl: A Package for Homotopy Continuation in Julia".** In: **Mathematical Software – ICMS 2018**. Ed. by J. H. Davenport, M. Kauers, G. Labahn, and J. Urban. Cham: Springer International Publishing, 2018, pp. 458–465.

[AFS19] C. Améndola, P. Friz, and B. Sturmfels. **"Varieties of signature tensors".** In: **Forum of Mathematics, Sigma** 7 (2019), e10.

[BES21] J. Berthomieu, C. Eder, and M. Safey El Din. **"msolve: A Library for Solving Polynomial Systems".** In: **Proceedings of the 2021 International Symposium on Symbolic and Algebraic Computation**. ISSAC '21. Virtual Event, Russian Federation: Association for Computing Machinery, 2021, pp. 51–58.

[BT18] P. Breiding and S. Timme. **"HomotopyContinuation.jl: A Package for Homotopy Continuation in Julia".** In: **Mathematical Software – ICMS 2018**. Ed. by J. H. Davenport et al. Cham: Springer International Publishing, 2018, pp. 458–465.

[Che58] K.-T. Chen. **"Integration of Paths–A Faithful Representation of Paths by Noncommutative Formal Power Series".** In: **Transactions of the American Mathematical Society** 89.2 (1958), pp. 395–407.

[CK19] J. Chen and J. Kileel. **"Numerical implicitization".** In: **The Journal of Software for Algebra and Geometry** 9 (2019).

[Lot+] F. Lotter et al. **PathSignatures: working with algebraic path signatures. Version 1.0.** A **Macaulay2** package.

[Lyo98] T. J. Lyons. **"Differential equations driven by rough signals.".** eng. In: **Revista Matemática Iberoamericana** 14.2 (1998), pp. 215–310.